ELSEVIER
Journal of Pure and Applied Algebra 177 (2003) 231–236
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Remarks on the frame envelope of a σ-frame

B. Banaschewski[a], P. Matutu[b,*]

[a] Department of Mathematics and Statistics, McMaster University, Ont., Canada
[b] Department of Mathematics, University of Stellenbosch, 7602 Matieland, South Africa

Received 1 December 2001
Communicated by D.S. Scott

Abstract

The familiar equivalence between σ-frames and σ-coherent frames, given by the frame envelopes of σ-frames, is shown to induce an equivalence between stably continuous σ-frames and stably continuous frames. Similarly, the analogue of the former for σ-biframes is proved to provide an equivalence between compact regular σ-biframes and compact regular biframes. As an application we obtain the equivalence between stably continuous σ-frames and compact regular σ-biframes due to Matutu as an easy consequence of its frame counterpart established earlier by Banaschewski and Brümmer. This provides an affirmative answer to a question posed by Dana Scott.

MSC: 54D30; 54D20; 06B35; 06D05

1. Preliminaries

A *frame* is a complete lattice L in which $x \wedge \bigvee S = \bigvee\{x \wedge s | s \in S\}$ for any $x \in L$ and $S \subseteq L$. A *frame homomorphism* $h : L \to M$ is a map between frames preserving finite meets (including the unit, or top element e) and arbitrary joins (including zero, or bottom element 0). We denote the resulting category of frames by **Frm**. For background on frames we refer to [4].

Taking countable subsets S and countable joins instead of arbitrary joins in the above definitions gives us *σ-frames* and *σ-frame homomorphisms*, and σ**Frm** will denote the corresponding category of σ-frames and σ-frame homomorphisms. For motivation and background regarding this category read [3]. For any σ-frame A, a *σ-ideal* of A is any

* Corresponding author.

0022-4049/03/$ - see front matter
PII: S0022-4049(02)00195-0

ideal which is closed under countable joins. If A is a σ-frame, then the collection of σ-ideals of A will be denoted by $\mathscr{H}A$. $\mathscr{H}A$ may be described as a quotient of the frame $\mathfrak{I}A$ of ideals of a σ-frame A [1]. Further, the embedding $A \to \mathscr{H}A$ taking $a \in A$ to the principal ideal $\downarrow a$ is the universal σ-frame homomorphism to frames.

In any frame L, $a \in L$ is called *Lindelöf* if $a \leqslant \bigvee X$ implies $a \leqslant \bigvee Y$ for some countable $Y \subseteq X \subseteq L$. $\mathrm{Lind}\, L$ will designate the set of Lindelöf elements of a frame L. A frame L is *σ-coherent* if $\mathrm{Lind}\, L$ is closed under finite meets and generates L. Taking as morphisms the frame homomorphisms which preserve Lindelöf elements we obtain the category σ**CohFrm**. Note that the correspondence $L \mapsto \mathrm{Lind}\, L$ is functorial on this category. Replacing Lindelöf elements by compact elements above we further obtain the category **CohFrm** of *coherent frames* and its morphisms. In [4] it has been established that the category $\mathbb{D}$ of bounded distributive lattices is equivalent to **CohFrm**. Replacing $\mathbb{D}$ by σ**Frm**, there is an analogous adjoint equivalence of σ**Frm** and σ**CohFrm** provided by the functors $\mathscr{H}$ and Lind.

A frame L is *compact* if $e = \bigvee X$ implies $e = \bigvee Y$ for some finite $Y \subseteq X \subseteq L$. In any complete lattice L, *a is way below b*, written $a \ll b$, if for any $X \subseteq L$ with $b \leqslant \bigvee X$, there exists a finite $Y \subseteq X$ with $a \leqslant \bigvee Y$. We say L is *continuous* if $a = \bigvee\{x \in L \mid x \ll a\}$ for all $a \in L$. Furthermore L is *stably continuous* if it is continuous, compact, and $a \ll b, c$ implies that $a \ll b \wedge c$. We are concerned with the category **StContFrm** of stably continuous frames, whose morphisms are frame homomorphisms which preserve the $\ll$ relation. The analogous notion for σ-frames where countable joins are considered instead of arbitrary joins gives us the σ-way below relation $\ll_\sigma$ and the category of **StContσFrm**.

The functor $\mathscr{H} : \sigma\mathbf{Frm} \to \sigma\mathbf{CohFrm}$ takes a σ-frame A to $\mathscr{H}A$ and morphisms $h : A \to B$ are mapped to $\mathscr{H}h : \mathscr{H}A \to \mathscr{H}B$, where for $J \in \mathscr{H}A$, $\mathscr{H}h(J)$ is the σ-ideal generated in B by $h(J)$. The functor $\mathrm{Lind} : \sigma\mathbf{CohFrm} \to \sigma\mathbf{Frm}$ takes a σ-coherent frame L to $\mathrm{Lind}\, L$, the σ-frame of its Lindelöf elements. It takes morphisms in σ**CohFrm** to the σ-frame homomorphisms induced on the Lindelöf elements. As a basic result we note from [1] that $\mathscr{H}$ and Lind form an adjoint category equivalence between σ**Frm** and σ**CohFrm**.

2. The equivalence between StContσFrm and StContFrm

Lemma 2.1. *Any stably continuous frame L is σ-coherent.*

Proof. We first show that $a \in L$ is Lindelöf iff $a = \bigvee\{x_n \mid x_n \ll a\}$.

$\Rightarrow$: Let $a = \bigvee\{x \mid x \ll a\}$ by continuity of L, but a is Lindelöf so there is a countable subset $\{x_n\}$ such that $a = \bigvee\{x_n \mid x_n \ll a\}$.

$\Leftarrow$: Consider $\bigvee\{x_n \mid x_n \ll a\} = a \leqslant \bigvee X$ with $X \subseteq L$, then for each n,

$x_n \ll a \leqslant \bigvee X$, hence there is a finite set $E_{n_i} \subseteq X$ such that $x_n \leqslant \bigvee E_{n_i}$, thus $a = \bigvee x_n \leqslant \bigvee \bigcup E_{n_i}$ and $\bigcup E_{n_i}$ is a countable subset of X. Thus a is Lindelöf.

If $a, b \in \mathrm{Lind}\, L$, then $a \wedge b = \bigvee\{a_n \wedge b_m \mid a_n \ll a,\ b_m \ll b\}$, hence $a_n \wedge b_m \ll a \wedge b$ so that $a \wedge b$ is Lindelöf. Lastly, $\mathrm{Lind}\, L$ generates L: Given $a \in L$, then $a = \bigvee\{x \mid x \ll a\}$, thus there is a sequence $\{c_n\}$ such that $c_0 = x \ll c_1 \ll c_2 \ll \cdots \ll c_n \ll c_{n+1} \ll \cdots \ll a$.

Note that $x \leqslant \bigvee c_n = c \leqslant a$, and c is Lindelöf, showing that $a = \bigvee \downarrow a \cap \operatorname{Lind} L$. Consequently L is σ-coherent. $\square$

Lemma 2.2. *Let L be a σ-coherent frame. For any $a, b \in \operatorname{Lind} L$, $a \ll b$ in $\operatorname{Lind} L$ iff $a \ll b$ in L.*

Proof. $\Leftarrow$: Is straightforward.

$\Rightarrow$: If $b \leqslant \bigvee X$ for any $X \subseteq L$, then $b \leqslant \bigvee Y$ for $Y = \cup\{\downarrow s \cap \operatorname{Lind} L \mid s \in X\}$, furthermore $b \leqslant \bigvee Z$ for a countable $Z \subseteq Y$, so that $a \leqslant \bigvee Z_0$ for a finite $Z_0 \subseteq Z$. Then there exists a finite $X_0 \subseteq X$ such that $z \leqslant x \in X_0$ for each $z \in Z_0$ and hence $a \leqslant \bigvee X_0$. $\square$

Corollary 2.3. *For any stably continuous frame L, $\operatorname{Lind} L$ is a stably continuous σ-frame.*

Proof. For any $a \in \operatorname{Lind} L$, $a = \bigvee\{a_n \mid a_n \ll a \text{ in } \operatorname{Lind} L\}$ by proof of Lemma 2.1. Since L is compact, so is $\operatorname{Lind} L$. If $a \ll b, c$ in $\operatorname{Lind} L$, then the relation also holds in L resulting in $a \ll b \wedge c$ by stably continuity of L. $\square$

Proposition 2.4. *$\mathscr{H}$ and Lind induce an equivalence between stably continuous σ-frames and stably continuous frames.*

Proof. We only have to show that $\mathscr{H}$ takes stably continuous σ-frames to stably continuous frames. This is the same as proving that if L is a σ-coherent frame such that $\operatorname{Lind} L$ is a stably continuous σ-frame, then L is stably continuous. In Lemma 2.2, it is shown that $a \ll b$ in $\operatorname{Lind} L$ implies that $a \ll b$ in L, thus L is continuous. Next L is compact. If $e = \bigvee X$ for any $X \subseteq L$ then also $e = \bigvee Y$, Y as in the proof of Lemma 2.2, hence $e = \bigvee Z$ for countable $Z \subseteq Y$ since L is Lindelöf, therefore $e = \bigvee Z_0$ for some finite $Z_0 \subseteq Z$, and then also $e = \bigvee X_0$ for some finite $X_0 \subseteq X$. Next, if $x \ll y$ in L, then there exist $a, b \in \operatorname{Lind} L$ such that $x \leqslant a \ll b \leqslant y$; thus if also $x \ll z$ so that $x \leqslant c \ll d \leqslant z$ with $c, d \in \operatorname{Lind} L$, then $x \leqslant a \wedge c \ll b \wedge d \leqslant y \wedge z$ and hence $x \ll y \wedge z$. Consequently, L is stably continuous. $\square$

3. The equivalence between KRσ2Frm and KR2Frm

We will need the following definitions.

A *σ-biframe* is a triple $M = (M_0, M_1, M_2)$ in which M_0 is a σ-frame and M_i, $i \in \{1, 2\}$ are sub σ-frames of M_0 such that $M_1 \cup M_2$ generates M_0, that is, each $m \in M_0$ is a countable join of finite meets from $M_1 \cup M_2$. A *σ-biframe map* $h : M \to N$ is a σ-frame map $h_0 : M_0 \to N_0$ such that $h(M_i) \subseteq N_i$, $i \in \{1, 2\}$. For $\{x, y\} \subseteq M_i$, $i \in \{1, 2\}$, we write $x \prec_i y$ (and say x is *i-rather below* y) iff there exists $c \in M_k$ ($i \neq k$) such that $x \wedge c = 0$ and $y \vee c = e$. M is *regular* if $x = \bigvee z_n$ where $z_n \prec_i x$ in M_i for all $x \in M_i$, $i \in \{1, 2\}$. The properties of $\prec_i$ are the same as those of $\prec$ in biframes and therefore omitted. M is *compact* if M_0 is compact. **KRσ2Frm** will denote the category of compact regular σ-biframes and σ-biframe homomorphisms.

The category of compact regular biframes and biframe homomorphisms will be denoted by **KR2Frm**. A biframe $L=(L_0,L_1,L_2)$ is *σ-coherent* if L_0 is σ-coherent, and L_i is generated by $L_i \cap \mathrm{Lind}\, L_0$ for $i \in \{1,2\}$. A biframe homomorphism $h: L \to M$ between σ-coherent biframes is *σ-coherent* if $h_0 : L_0 \to M_0$ is σ-coherent. Let σ**Coh2Frm** be the resulting category.

We prove the equivalence of σ**2Frm** with σ**Coh2Frm** using that of σ**Frm** with σ**CohFrm**.

Lemma 3.1. *For the functor* $\mathscr{H}$: σ**Frm** → σ**CohFrm**, $f: A \to B$ *is onto* (*respectively*, *one–one*) *in* σ**Frm** *if and only if* $\mathscr{H}f : \mathscr{H}A \to \mathscr{H}B$ *is onto* (*respectively*, *one–one*) *in* σ**CohFrm**.

Proof. Firstly we show the onto equivalence.

⇒: Let $J \in \mathscr{H}B$, then $I = f^{-1}(J) \in \mathscr{H}A$ and $f(I)=J$, hence $\mathscr{H}f(I)=J$.

⇐: If $c \in B$, then $\downarrow c \in \mathscr{H}B$ and there exists $J \in \mathscr{H}A$ such that $\mathscr{H}f(J) = \downarrow c$, but $\mathscr{H}f(J) = \bigcup\{\downarrow f(a) \mid a \in J\} = \downarrow c$, so that $c = f(a)$ for some $a \in J \subseteq A$.

Next we show the one–one equivalence.

⇐: Consider $f: A \to B$, with $\mathscr{H}f : \mathscr{H}A \to \mathscr{H}B$ injective and suppose that $f(a_1) = f(a_2)$. Then $\downarrow f(a_1) = \downarrow f(a_2)$, hence $\mathscr{H}f(\downarrow a_1) = \mathscr{H}f(\downarrow a_2)$ and thus $\downarrow a_1 = \downarrow a_2$, showing that $a_1 = a_2$. Thus $f: A \to B$ is injective.

⇒: If $f: A \to B$ is injective and $J_1, J_2 \in \mathscr{H}A$ then we show that $\mathscr{H}f(J_1) = \mathscr{H}f(J_2)$ implies that $J_1 = J_2$. Let $a \in J_1$ then $\downarrow f(a) \subseteq \mathscr{H}f(J_1)$, thus $\downarrow f(a) \subseteq \mathscr{H}f(J_2)$, hence $f(a) \leqslant f(b)$ for some $b \in J_2$, but f is one–one, so that $a \leqslant b$, consequently $a \in J_2$. This shows that $J_1 \subseteq J_2$ and hence $J_1 = J_2$ by symmetry. □

Next we recall from [6] the functor $\mathscr{H}$: σ**2Frm** → σ**Coh2Frm** which takes each σ-biframe A to the σ-coherent biframe $\mathscr{H}A$ where $(\mathscr{H}A)_0 = \mathscr{H}A_0$ and

$$(\mathscr{H}A)_i = \{J \in \mathscr{H}A_0 \mid J \text{ generated by } J \cap A_i\} \quad (i = 1,2)$$

and any $f: A \to B$ in σ**2Frm** to $\mathscr{H}f : \mathscr{H}A \to \mathscr{H}B$ such that $\mathscr{H}f(J)$ is the σ-ideal generated by $f(J)$ in B_0 for each $J \in \mathscr{H}A_0$. Further, define Lind : σ**Coh2Frm** → σ**2Frm** as

$$\mathrm{Lind}\, L = (\mathrm{Lind}\, L_0, L_1 \cap \mathrm{Lind}\, L_0, L_2 \cap \mathrm{Lind}\, L_0)$$

for each σ-coherent biframe L, with the obvious action on the maps.

Proposition 3.2. *These functors* $\mathscr{H}$ *and* Lind *provide an adjoint equivalence between* σ**2Frm** *and* σ**Coh2Frm**.

Proof. First we note that a pair of maps $A_1 \to A_0 \leftarrow A_2$ in σ**Frm** is jointly onto iff the corresponding pair $\mathscr{H}A_1 \to \mathscr{H}A_0 \leftarrow \mathscr{H}A_2$ in σ**Coh Frm** is jointly onto. Clearly, $A_1 \to A_0 \leftarrow A_2$ is jointly onto iff $A_1 \oplus A_2 \to A_0$ is onto which holds iff $\mathscr{H}(A_1 \oplus A_2) \to \mathscr{H}A_0$ is onto by Lemma 3.1, and since $\mathscr{H}$: σ**Frm** → **Frm** as left adjoint, preserves coproducts this proves the claim. Further, again by Lemma 3.1, $A_i \to A_0$ is one–one iff $\mathscr{H}A_i \to \mathscr{H}A_0$ is one–one, for $i \in \{1,2\}$. Consequently, the functors $\mathscr{H}$ and

Lind between σ**Frm** and σ**CohFrm** induce an equivalence between the categories of diagrams $A_1 \to A_0 \leftarrow A_2$ in σ**Frm** and $L_1 \to L_0 \leftarrow L_2$ in σ**CohFrm** in which the maps are one–one and jointly onto. Moreover, replacing embeddings by images and identical embeddings, this equivalence takes the diagram $A_1 \to A_0 \leftarrow A_2$ for a σ-biframe A to

$$\mathrm{Im}(\mathscr{H}A_1 \to \mathscr{H}A_0) \to \mathscr{H}A_0 \leftarrow \ \mathrm{Im}(\mathscr{H}A_2 \to \mathscr{H}A_0),$$

which is exactly the diagram for the biframe $\mathscr{H}A$ described above.

On the other hand, the diagram $L_1 \to L_0 \leftarrow L_2$ for a σ-coherent biframe L is taken to $\mathrm{Lind}\, L_1 \to \ \mathrm{Lind}\, L_0 \leftarrow \mathrm{Lind}\, L_2$ and since $\mathrm{Lind}\, L_i = L_i \cap \mathrm{Lind}\, L_0$ for $i \in \{1,2\}$ by the definition of σ-coherent biframes, this is the diagram for the above $\mathrm{Lind}\, L$.

In all, then, the present functors $\mathscr{H}$ and Lind provide an equivalence, as claimed. □

Proposition 3.3. *$\mathscr{H}$ and* Lind *induce an equivalence between* **KRσ2Frm** *and* **KR2Frm**.

Proof. For any compact regular σ-biframe A, $(\mathscr{H}A)_0 = \mathscr{H}A_0$ is compact since A_0 is compact [1] and hence $\mathscr{H}A$ is compact. Further, if $a = \bigvee a_n$ and $a_n \prec_i a$ in L_i $(i = 1,2)$ then $\downarrow a = \bigvee \downarrow a_n$ and $\downarrow a_n \prec_i \downarrow a$ in $(\mathscr{H}A)_i$, and since the $\downarrow a$ generate $(\mathscr{H}A)_i$ this proves the regularity of $\mathscr{H}A$.

It remains to show that any compact regular biframe L is σ-coherent. Now, each L_i for $i \in \{0,1,2\}$ is stably continuous: $\ll$ is $\prec$ in L_0 while $\ll$ on L_i for $i \in \{1,2\}$ is $\prec_i$, and hence each L_i is σ-coherent by Lemma 2.1. Moreover, for any $a \in L_i$, $i \in \{1,2\}$ the Lindelöf elements $c \leqslant a$ obtained in the proof of Lemma 2.1 as $\bigvee c_n$ where $c_0 \ll c_1 \ll c_2 \ll \cdots \leqslant a$ actually belong to $\mathrm{Lind}\, L_0$. It follows that L_i is generated by $L_i \cap \mathrm{Lind}\, L_0$, showing that L is σ-coherent. □

As an application we now have the following diagram of equivalences:

$$\begin{array}{ccc} \mathbf{KR\sigma 2Frm} & \xrightarrow{\mathscr{H}} & \mathbf{KR2Frm} \\ {\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ \mathbf{StCont\sigma Frm} & \xrightarrow{\mathrm{Lind}} & \mathbf{StContFrm} \end{array}$$

where $\mathscr{H}$ comes from Proposition 3.3, Lind from Proposition 2.4, F is the equivalence of [2] taking each compact regular biframe L to its first part L_1, and $G = \mathrm{Lind}\, F\mathscr{H}$.

Now, G takes each compact regular σ-biframe A to $\mathrm{Lind}\,(\mathscr{H}A)_1$ where

$$(\mathscr{H}A)_1 = \{J \in \mathscr{H}A_0 \mid J \text{ generated by } J \cap A_1\}$$

and the Lindelöf elements of this are readily identified as the $\downarrow a$ for $a \in A_1$ so that $GA \cong A_1$, obviously natural in A. Thus we have a new proof of the result of [5]:

For σ-biframes, the functor taking first parts induces an equivalence **KRσ2Frm** $\to$ **StContσFrm**.

We note that the question whether this could be reduced to the equivalence F by means of suitable transitions between σ-frames and frames was raised by Dana Scott at the category conference in Coimbra, summer 1999.

Acknowledgements

The authors are grateful for financial assistence from the following:
the Category Theory and Topology Research Group at the University of Cape Town for supporting a visit by B. Banaschewski from September 1999 to January 2000 during which part of this work was done, the Natural Sciences and Engineering Research Council of Canada for ongoing support of B. Banaschewski in the form of a research grant, and the National Research Foundation of South Africa for support of P. Matutu in the form of the REDIBA grant in 2001.

References

[1] B. Banaschewski, The frame envelope of a σ-frame, Quaestiones Math. 16 (1993) 51–60.
[2] B. Banaschewski, G.C.L. Brümmer, Stably continuous frames, Math. Proc. Cambridge Philos Soc. 104 (1988) 7–19.
[3] B. Banaschewski, C.R.A. Gilmour, Stone-Čech. Compactification and dimension theory for regular σ-frames, J. London Math. Soc. 39 (1989) 1–8.
[4] P.T. Johnstone, Stone Spaces, Cambridge University Press, Cambridge, New York, Melbourne, 1982.
[5] P. Matutu, Stably continuous σ-frames, Quaestiones Math. 24 (2001) 201–211.
[6] P. Matutu, The cozero part of a biframe, Kyungpook Math. J. 42(2) (2002), to appear.